

COSMOLOGICAL DYNAMICS OF MODIFIED GRAVITY WITH A NON-MINIMAL CURVATURE-MATTER COUPLING

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We perform a phase space analysis of a non-minimally coupled modified gravity theory with the Lagrangian density of the form $\frac{1}{2}f_1(R) + [1 + \lambda f_2(R)]\mathcal{L}_m$, where $f_1(R)$ and $f_2(R)$ are arbitrary functions of the curvature scalar R and \mathcal{L}_m is the matter Lagrangian density. We apply the dynamical system approach to this scenario in two particular models. In the first model we assume $f_1(R) = 2R$ with a general form for $f_2(R)$ and set favorable values for effective equation of state parameter which is related to the several epochs of the cosmic evolution and study the critical points and their stability in each cosmic eras. In the second case, we allow the $f_1(R)$ to be an arbitrary function of R and set $f_2(R) = 2R$. We find the late time attractor solution for the model and show that this model has a late time accelerating epoch and an acceptable matter era.

Keywords: Dynamical system analysis; Modified gravity; Non-minimal coupling

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1. Introduction

Recent observational data suggest that our universe is undergoing an accelerating phase of expansion [1-4]. One way to explain the cosmic speedup within the framework of general relativity is adding a mysterious component to the matter content of the universe which is dubbed as dark energy. Several candidates to the dark energy are proposed such as cosmological constant, the scalar fields, Chaplygin gas and so on (see [5] and references therein). Another popular approach to describe the accelerating expansion of the universe is to assume a modification to the general theory of relativity on cosmological scales. In this respect the so-called $f(R)$ modified gravity theory has attracted a lot of attention [6-9]. In these theories, the curvature scalar R is replaced by a generic function of R in the action. For a review of $f(R)$ modified gravity and its cosmological implications see [10-12]. Recently,

an extension of $f(R)$ theories has been proposed in [13] which there is an explicit coupling between matter Lagrangian density and the curvature scalar (see also [14] and [15]). This model leads to considerable cosmological implications which some of them are: the energy exchange between the matter fields and the curvature and deviation from geodesic motion [13], the mimicking of dark matter by leading to the flattening of the galaxy rotation curves [16, 17], the modelling of the cosmic speed up at late times [18] and the reheating scenario after inflation [19]. The cosmological perturbations and the matter density perturbations of this scenario are studied in [20] and [21] respectively.

The aim of this paper is to study the dynamics of the modified gravity theory with a non-minimal coupling between matter and geometry via a phase space analysis approach. The strategy is to rewrite Einstein's field equations for cosmological models in terms of an autonomous system of ordinary differential equations (ODE) [22-29]. The paper is organized as follows. In Section 2 we introduce the non-minimal modified gravity scenario and its gravitational field equations. Using the flat Friedman-Robertson-Walker metric with a perfect fluid form of the stress energy tensor, we obtain the generalized Friedmann equations of the scenario. In order to study the cosmological dynamics of this model, we consider two particular cases in the rest of this paper. In section 3 we set $f_1(R) = 2R$ and a general form for $f_2(R)$ and apply a phase-space analysis approach to obtain the critical points and their stability in several eras of the cosmic evolution. Setting $f_2(R) = 2R$ and a power law form for $f_1(R)$, in section 4, we study the dynamical system and analyze the nature of the resulted critical points of this system. Finally, section 5 is devoted to conclusions.

2. The Equations of Motion

The action of the modified gravity model with a non-minimal coupling with matter is given by [13]

$$S = \int \left[\frac{1}{2} f_1(R) + [1 + \lambda f_2(R)] \mathcal{L}_m \right] \sqrt{-g} d^4x. \quad (1)$$

where \mathcal{L}_m is the matter Lagrangian density, $f_1(R)$ and $f_2(R)$ are arbitrary functions of the curvature scalar R and λ is a coupling parameter. Varying the action (1) with respect to the metric $g_{\mu\nu}$ yields the following gravitational field equations [13]

$$[f_1'(R) + 2\lambda f_2'(R) \mathcal{L}_m] R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} f_1(R) = (\nabla_\mu \nabla_\nu - g_{\mu\nu} \square)(f_1'(R) + 2\lambda f_2'(R) \mathcal{L}_m) + [1 + \lambda f_2(R)] T_{\mu\nu}, \quad (2)$$

where a prime denotes the derivative with respect to the curvature scalar. $T_{\mu\nu}$ is the stress-energy tensor and is related to the Lagrangian density of the matter as follows

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} \mathcal{L}_m)}{\delta g^{\mu\nu}}. \quad (3)$$

Taking the trace of Eq.(2) leads us to

$$[f'_1(R) + 2\lambda f'_2(R)\mathcal{L}_m]R - 2f_1(R) = -3\Box[f'_1(R) + 2\lambda f'_2(R)\mathcal{L}_m] + [1 + \lambda f_2(R)]T \quad (4)$$

where T is the trace of the stress-energy tensor. We assume that the matter content of the universe is described by a perfect fluid with an energy momentum tensor

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + pg_{\mu\nu}, \quad (5)$$

where u^μ is the four velocity of the fluid in comoving coordinates. It is worthwhile to notice that in the presence of a non-minimal coupling between matter and curvature, the energy-momentum tensor is not covariantly conserved which implies that the motion of a point like particle is non-geodesic [13].

Now, we define an effective energy-momentum tensor in equation (2) thus the field equation recast in the form of the standard Einsteins equation

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = T_{\mu\nu}^{(eff)}, \quad (6)$$

where $T_{\mu\nu}^{(eff)}$ is defined by

$$T_{\mu\nu}^{(eff)} = T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(c)}, \quad (7)$$

where we have defined $T_{\mu\nu}^{(m)}$ and $T_{\mu\nu}^{(c)}$ as follows

$$T_{\mu\nu}^{(m)} = \frac{1}{f'_1 + 2\lambda f'_2\mathcal{L}_m} T_{\mu\nu}, \quad (8)$$

$$T_{\mu\nu}^{(c)} = \frac{1}{f'_1 + 2\lambda f'_2\mathcal{L}_m} \left[\frac{1}{2}(f_1 - f'_1 R)g_{\mu\nu} - \lambda f'_2 R\mathcal{L}_m g_{\mu\nu} + (\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box)(f'_1 + 2\lambda f'_2\mathcal{L}_m) + \lambda f_2 T_{\mu\nu} \right] \quad (9)$$

respectively. We assume that the matter Lagrangian density is $\mathcal{L}_m = -\rho_m$ (for a discussion about the possibility of other choices for the matter Lagrangian see for instance Ref. 30 and references therein). Using the flat Friedman-Robertson-Walker metric $ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2$, the equations of motion (2) lead to the modified Friedmann equations as follows

$$3H^2 = T_{tt}^{(m)} + T_{tt}^{(c)}, \quad (10)$$

$$-\dot{a}^2 - 2\ddot{a}a = T_{rr}^{(m)} + T_{rr}^{(c)}. \quad (11)$$

where $H = \dot{a}/a$ is the Hubble parameter. The latter equation can be reduced to

$$-\frac{\dot{a}^2}{a^2} - \frac{2\ddot{a}a}{a^2} = p_m + p_c \quad (12)$$

where p_m and p_c are the pressure of the matter and curvature fluid respectively. In the case of dust matter with $p_m = 0$ and regarding to the fact $R = 6(2H^2 + H\dot{H})$, equations (10) and (11) can be rewritten as

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$$3H^2 = \frac{R}{2} - \frac{f_1}{2(f_{1,R} - 2\lambda f_{2,R}\rho_m)} - \frac{3H^2 f_{1,RR}R'}{(f_{1,R} - 2\lambda f_{2,R}\rho_m)} + \frac{6H^2 \lambda f_{2,RR}R' \rho_m}{(f_{1,R} - 2\lambda f_{2,R}\rho_m)} + \frac{6H^2 \lambda f_{2,R}\rho'_m}{(f_{1,R} - 2\lambda f_{2,R}\rho_m)} + \frac{\lambda f_{2,R}\rho_m}{(f_{1,R} - 2\lambda f_{2,R}\rho_m)} + \frac{\rho_m}{(f_{1,R} - 2\lambda f_{2,R}\rho_m)} \quad (13)$$

and

$$p_c = - \left(\frac{R}{3} - H^2 \right). \quad (14)$$

respectively, where $f_{i,R} \equiv df_i/dR$, $f_{i,RR} \equiv d^2 f_i/dR^2$ and $f_{i,RRR} \equiv d^3 f_i/dR^3$. Here we have defined

$$' = \frac{d}{d \ln a} \equiv \frac{d}{dN} = \frac{1}{H} \frac{d}{dt} \quad (15)$$

and ρ_m represent the matter energy density which is conserved by virtue of the continuity equation via the relation

$$\dot{\rho}_m + 3H\rho_m = 0. \quad (16)$$

Since a comprehensive study of the effect of a nontrivial $f_1(R)$ and $f_2(R)$ is complicated, in the rest of this paper, we take two ansatzes. In the first approach, we set $f_1(R) = 2R$ and $f_2(R) = f(R)$ and in the second case we set $f_1(R) = f(R)$ and $f_2(R) = 2R$ to study the dynamical behavior of the system.

3. Autonomous Equations For the Case $f_1(R) = 2R$ And $f_2(R) = f(R)$

In the case $f_1(R) = 2R$ and $f_2(R) = f(R)$, the Friedmann equations (13) and (14) can be rewritten as

$$3H^2 = \frac{1}{2 - 2\lambda f_{,R}\rho_m} \rho_m + \frac{1}{2 - 2\lambda f_{,R}\rho_m} \left\{ -\lambda f_{,R}R\rho_m + 6\lambda H^2 f_{,RR}R'\rho_m - 18\lambda H^2 f_{,R}\rho_m + \lambda f\rho_m \right\} \quad (17)$$

and

$$- \left(\frac{R}{3} - H^2 \right) = \frac{-\lambda}{2 - 2\lambda f_{,R}\rho_m} \left\{ 2H^2 R'^2 f_{,RRR}\rho_m + 2f_{,RR}HH'R'\rho_m + 2f_{,RR}H^2 R''\rho_m + 6H^2 f_{,RR}R'\rho_m - f_{,R}R\rho_m - 6HH'f_{,R}\rho_m - 12f_{,RR}H^2 R'\rho_m \right\} \quad (18)$$

respectively. We can express the generalized friedmann equations (17) and (18) in an autonomous system of the first ODEs to study the cosmological dynamics of the model ^{22,23,25,28,29}. For this purpose, we express the generalized friedmann equation (17) in a dimensionless form as follows

$$1 = \frac{1}{6} \frac{\rho_m}{H^2(1 - \lambda f_{,R}\rho_m)} + \frac{\lambda f_{,RR}R'\rho_m}{1 - \lambda f_{,R}\rho_m} - \frac{1}{6} \frac{\lambda f_{,R}R\rho_m}{H^2(1 - \lambda f_{,R}\rho_m)} - 3 \frac{\lambda f_{,R}\rho_m}{1 - \lambda f_{,R}\rho_m} + \frac{1}{6} \frac{\lambda f\rho_m}{H^2(1 - \lambda f_{,R}\rho_m)}. \quad (19)$$

We define the dimensionless variables x , y , z and v as

$$x = \frac{\lambda f_{,RR} R' \rho_m}{1 - \lambda f_{,R} \rho_m}, \quad (20)$$

$$y = -\frac{1}{6} \frac{\lambda f_{,R} R \rho_m}{H^2 (1 - \lambda f_{,R} \rho_m)}, \quad (21)$$

$$z = -3 \frac{\lambda f_{,R} \rho_m}{1 - \lambda f_{,R} \rho_m}, \quad (22)$$

and

$$v = \frac{1}{6} \frac{\lambda f \rho_m}{H^2 (1 - \lambda f_{,R} \rho_m)} \quad (23)$$

respectively. By defining the density parameter $\Omega = \frac{1}{6} \frac{\rho_m}{H^2 (1 - \lambda f_{,R} \rho_m)}$, the equation (19) takes the form

$$\Omega = \frac{1}{6} \frac{\rho_m}{H^2 (1 - \lambda f_{,R} \rho_m)} = 1 - x - y - z - v. \quad (24)$$

Using equ. (18), differentiating x with respect to N leads to

$$x' = -3 \frac{xy}{z} + 6 \frac{y}{z} + x^2 + 2x - 6y + 2z + xz - 1. \quad (25)$$

Similarly, differentiating the equation (21) with respect to N results

$$y' = -3 \frac{xy}{mz} - 3 \frac{xy}{z} - 6 \frac{y^2}{z} + xy + yz + y, \quad (26)$$

where

$$m = \frac{\ln F}{\ln R} = \frac{R f_{,RR}}{f_{,R}}. \quad (27)$$

Also, differentiating (22) and (23) leads to

$$z' = z^2 + xz - 3x - 3z \quad (28)$$

and

$$v' = 3 \frac{xy}{mz} - 6 \frac{yv}{z} + v + xv + zv \quad (29)$$

respectively. The effective equation of state parameter is defined as

$$w_{eff} = -1 - \frac{2}{3} \frac{H'(N)}{H(N)}. \quad (30)$$

using equation (15) and (18), we can rewrite this relation in the following form

$$w_{eff} = -2 \frac{y}{z} + \frac{1}{3}. \quad (31)$$

Now our dynamical system depends on $m(f(R))$. We can eliminate m from the system by solving Equ.(26) for $\frac{xy}{zm}$ and substituting in (29) with definition $s = \frac{y}{z}$,

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consequently equation (29) can be rewritten as $v' = -sz' - 3\frac{xy}{z} - 6\frac{y^2}{z} + xy + yz + y - 6\frac{yv}{z} + v + xv + zv$. If we substitute z' from (28), then our dynamical system is given by as follows

$$x' = -3sx + 6s + 2x - 6sz + 2z + x^2 + xz - 1, \quad (32)$$

$$z' = z^2 + xz - 3x - 3z, \quad (33)$$

$$v' = 4sz - 6s^2z - 6sv + v + xv + zv. \quad (34)$$

In order to study the dynamics of the system of equations (32), (33) and (34), we set $x' = 0$, $z' = 0$ and $v' = 0$ and find the critical points and their stability in each one of the cosmic eras. The critical points of the scenario and the corresponding Ω are summarized in table. 1. A general eigenvalues matrix is obtained as below

$$\begin{bmatrix} \frac{1}{2}(-1 + 3z + 3x - 3s + 1\sqrt{1 - 2z - 2x + 42s + z^2 + 2xz - 18sz + x^2 - 6sx + 9s^2}) \\ \frac{1}{2}[-1 + 3z + 3x - 3s - 1\sqrt{1 - 2z - 2x + 42s + z^2 + 2xz - 18sz + x^2 - 6sx + 9s^2}] \\ -6s + 1 + x + z \end{bmatrix}$$

Table 1. The critical points of the dynamical system (32), (33) and (34).

point	x	y	z	v	Ω
A	$\frac{1-6s}{3s}$	$-\frac{1-6s}{3}$	$-\frac{1-6s}{3s}$	$-2s + \frac{4}{3}$	0
B	$\frac{3}{2}s - \frac{5}{2} + \frac{1}{2}\sqrt{9s^2 + 18s + 5}$	$3s$	3	$\frac{12s-18s^2}{\frac{9}{2}s - \frac{3}{2} - \frac{1}{2}\sqrt{9s^2 + 18s + 5}}$	$-\frac{12s-18s^2}{\frac{9}{2}s - \frac{3}{2} - \frac{1}{2}\sqrt{9s^2 + 18s + 5}} - \frac{9}{2}s + \frac{1}{2} - \frac{1}{2}\sqrt{9s^2 + 18s + 5}$
C	$\frac{3}{2}s - \frac{5}{2} - \frac{1}{2}\sqrt{9s^2 + 18s + 5}$	$3s$	3	$\frac{12s-18s^2}{\frac{9}{2}s - \frac{3}{2} + \frac{1}{2}\sqrt{9s^2 + 18s + 5}}$	$-\frac{12s-18s^2}{\frac{9}{2}s - \frac{3}{2} + \frac{1}{2}\sqrt{9s^2 + 18s + 5}} - \frac{9}{2}s + \frac{1}{2} + \frac{1}{2}\sqrt{9s^2 + 18s + 5}$

The corresponding effective equation of state is

$$w_{eff} = -2s + \frac{1}{3}. \quad (35)$$

One can set favorable values for w_{eff} which are related to the several eras of the cosmic evolution and obtain the corresponding s . We study the system in which the universe undergoes through the radiation era ($w_{eff} = \frac{1}{3}$), matter era ($w_{eff} = 0$) and de Sitter era ($w_{eff} = -1$) to mimic a de Sitter late time cosmology²⁹.

The critical points in radiation, matter and de Sitter eras and their stability in each eras are shown in table 2. It is clear that for the radiation era, we have two critical points which one of them is a saddle point and the other is unstable. For the matter era, there exist three critical points which one of them is a saddle point which corresponds to the usual matter era. In the de Sitter era we also find three critical points which one of them is a stable point corresponding to the late time acceleration of the cosmic evolution. In Fig. 1 (left panel), we depict the corresponding $x - z - v$ phase space behavior for the matter dominated era. As this figure shows, the critical point $B(-0.81, 3, -0.68)$ is an unstable point and the point $C(-3.68, 3, 2.18)$ is a stable point because the trajectories of the phase space are attracted by this critical

point. On the other hand, the point $A(0, 0, 1)$ is a saddle point which is a suitable point to the matter dominated phase. The $x - z - v$ phase space behavior for the de Sitter dominated era is plotted in the right panel of the Fig. 1. This figure shows that the critical point $B(-1.5, 1.5, 0)$ is a saddle point and the phase space trajectories are passed from this point and attracted by the stable critical point $C(-3.79, 3, 0)$. So the critical point C is a suitable point for the de Sitter dominated phase corresponding to the late time acceleration of the universe.

Table 2. The critical points of the system of equations (32),(33) and (34) and their stability in each one of the three eras. Source points (stable points) have only negative eigenvalues, saddle points have mixed sign eigenvalues and sink points (unstable points) have positive eigenvalues.

Era	x	y	z	v	Ω	w_{eff}	Eigenvalues
Radiation	-1.38	0	3	0	-0.62	$\frac{1}{3}$	(2.62, 2.24, 1.62)
Radiation	-3.61	0	3	0	1.61	$\frac{1}{3}$	(0.39, -0.61, -2.22)
Matter	0	0	0	1	0	0	(0, 0.68, -2.1)
Matter	-0.81	0.5	3	-0.68	-1	0	(2.19, 2.88, 2.19)
Matter	-3.68	0.5	3	2.18	-1	0	(-0.69, -0.69, -2.88)
de Sitter	-1.5	1	1.5	0	0	-1	(0.79, -3.79, -3)
de Sitter	0.79	2	3	0	-4.79	-1	(4.58, 3.79, 79)
de Sitter	-3.79	2	3	0	-0.21	-1	(-0.79, -4.58, -3.79)

4. Autonomous Equations For the Case $f_1(R) = f(R)$ And $f_2(R) = 2R$

In this section, we allow that $f_1(R)$ to be an arbitrary function of R and set $f_2(R) = 2R$, for little values of the coupling parameter λ , the field equation (13) can be approximated as

$$1 = -\frac{f}{6H^2 f_R} - \frac{2\lambda f \rho_m}{6H^2 f_R^2} + \frac{R}{6H^2} - \frac{2\lambda f_{RR} R' \rho_m}{f_R^2} - \frac{f_{RR} R'}{f_R} - \frac{6\lambda \rho_m}{f_R} + \frac{\lambda R \rho_m}{3H^2 f_R} + \frac{2\lambda \rho_m^2}{3H^2 f_R^2} + \Omega_m \quad (36)$$

where

$$\Omega_m = \frac{\rho_m}{3H^2 f_R} \quad (37)$$

is the density parameter of the matter field and prime denote a derivative with respect to $\ln a$. In the rest of this section, we consider a power-law model for $f(R)$,

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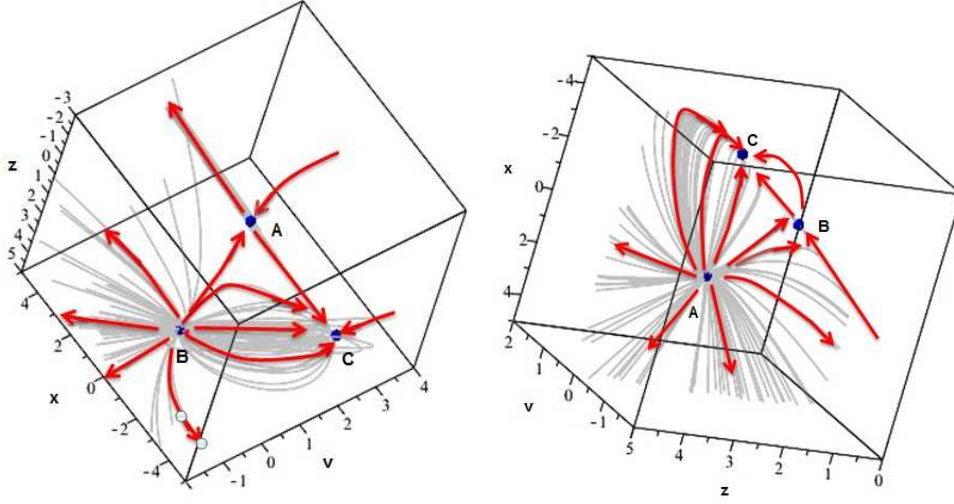


Fig. 1. The 3D phase space $x - v - z$ behavior for the matter era (left panel) and the de Sitter era (right panel).

i.e. $f(R) = R^n$, then the generalized Friedman equation (36) takes the form

$$1 = \left(\frac{n-1}{n}\right) \frac{R}{6H^2} + \left(\frac{n-1}{n^2}\right) \frac{2\lambda\rho_m}{3H^2 R^{n-2}} - (n-1) \frac{R'}{R} - \frac{12\lambda\rho_m}{nR^{n-1}} - 4\left(\frac{n-1}{n}\right) \frac{R'\lambda\rho_m}{R^n} + \frac{4\lambda\rho_m^2}{3H^2 n^2 R^{2n-2}} + \Omega_m \quad (38)$$

To study the phase space analysis of this scenario, we shall introduce the following dimensionless variables

$$x_1 = -(n-1) \frac{R'}{R}, \quad (39)$$

$$x_2 = -\frac{12\lambda\rho_m}{nR^{n-1}}, \quad (40)$$

$$x_3 = \left(\frac{n-1}{n}\right) \frac{R}{6H^2}, \quad (41)$$

$$x_4 = \left(\frac{n-1}{n^2}\right) \frac{2\lambda\rho_m}{3H^2 R^{n-2}}, \quad (42)$$

$$x_5 = \frac{4\lambda\rho_m^2}{3H^2 n^2 R^{2n-2}}, \quad (43)$$

$$x_6 = -4\left(\frac{n-1}{n}\right) \frac{R'\lambda\rho_m}{R^n}, \quad (44)$$

Similarly to the previous section, the equations of motion for the autonomous equations (39)-(44) are obtained as follows

$$x'_1 = \frac{3}{3-x_2} \left(x^2 - \frac{n}{n-1} x_1 x_3 - \frac{n}{n-1} x_2 x_3 - \frac{n}{n-1} x_3 x_6 + x_1 x_6 - \frac{n-3}{n-1} x_3 + 2x_2 - x_1 + \frac{3}{n-1} x_4 - x_6 - 1 \right), \quad (45)$$

$$x'_2 = -3x_2 + x_1 x_2, \quad (46)$$

$$x'_3 = -\frac{1}{n-1} x_1 x_3 - 2\frac{n}{n-1} x_3^2 + 4x_3, \quad (47)$$

$$x'_4 = -2\frac{n}{n-1} x_4 x_3 + x_4 - \frac{1}{3} \frac{n-2}{n-1} x_1 x_2 x_3, \quad (48)$$

$$x'_5 = -2x_5 - 2\frac{n}{n-1} x_3 x_5 + \frac{2n-2}{n-1} x_5 x_1 \quad (49)$$

and

$$x'_6 = -\frac{x_2}{3-x_2} \left(x^2 - \frac{n}{n-1} x_1 x_3 - \frac{n}{n-1} x_2 x_3 - \frac{n}{n-1} x_3 x_6 + x_1 x_6 - \frac{n-3}{n-1} x_3 + 2x_2 - x_1 + \frac{3}{n-1} x_4 - x_6 - 1 \right) - 3x_6 + x_1 x_6. \quad (50)$$

With the constraint equation (38), the density parameter of the matter field can be obtained as $\Omega_m = \frac{\rho_m}{3H^2 n R^{n-1}} = 1 - x_1 - x_2 - x_3 - x_4 - x_5 - x_6$. Also, with the definition (30), the effective equation of state parameter is $w_{eff} = -\frac{2}{3} \frac{n}{n-1} x_3 + \frac{1}{3}$. In order to investigate the dynamics which is implied by the equations of motions, we set $x'_i = 0$ (i=1-8) and find the critical points which are shown in table 3. Where we have defined $A = \sqrt{84n^4 - 252n^3 + 253n^2 - 90n + 9}$.

Among the critical points of the dynamical system, we discuss some important cases of them as

Table 3. The critical points of the dynamical system (45-50).

Points	Coordinates $(x_1, x_2, x_3, x_4, x_5, x_6)$	Ω_m	w_{eff}
P_1	$(0, \frac{1}{3}, 0, 0, 0, -\frac{1}{3})$	0	$\frac{1}{3}$
P_2	$(1.61, 0, 0, 0, 0, 0)$	-0.61	$\frac{1}{3}$
P_3	$(-0.61, 0, 0, 0, 0, 0)$	1.61	$\frac{1}{3}$
P_4	$(3, \frac{-6n^2+30n-21}{4n-7}, \frac{4n-7}{2n}, \frac{2n^2-10n+7}{2n}, 0, \frac{6n^2-30n+21}{4n-7})$	$-n+1$	$\frac{2-n}{n-1}$
P_5	$(0, \frac{3(n^2-3n+2)}{n^2-3n+4}, \frac{2(n-1)}{n}, \frac{2(-n^3+5n^2-8n+4)}{n(3n^2-5n+4)}, 0, \frac{3(-n^2+3n-2)}{3n^2-5n+4})$	$\frac{-n^2+n+2}{3n^2-5n+4}$	-1
P_6	$(3n-3, 0, \frac{1}{2} \frac{n-1}{n}, \frac{1}{2} \frac{-23n^2+21n^3-6n^4+7n+1}{n}, 0, 0)$	$\frac{17}{2}n - \frac{21}{2}n^2 + 3n^3$	0
P_7	$(\frac{6n^2-7n+3-A}{2n(2n-1)}, 0, \frac{16n^3-30n^2+15n-3+A}{4n^2(2n-1)}, 0, 0, 0)$	$\frac{-10n^2+15n-3+A}{4n^2}$	$\frac{24n^2-12n^3-13n+3-A}{6n(n-1)(2n+1)}$
P_8	$(\frac{6n^2-7n+3+A}{2n(2n-1)}, 0, \frac{16n^3-30n^2+15n-3-A}{4n^2(2n-1)}, 0, 0, 0)$	$\frac{-10n^2+15n-3-A}{4n^2}$	$\frac{24n^2-12n^3-13n+3+A}{6n(n-1)(2n+1)}$

follows

- The Point P_4

This point for $n \rightarrow 1^+$ and $n \rightarrow 1^-$ has $\Omega_{de} \approx 1$ with $w_{eff} \rightarrow +\infty$ and $w_{eff} \rightarrow -\infty$ respectively which are not cosmologically acceptable. The point P_4 for $n = 0$ has $\Omega_m = 1$ but some of the critical points disappear for that so it may be interesting to consider the scenario with $n \rightarrow 0^+$ and $n \rightarrow 0^-$ which both have $w_{eff} \approx -2$ which are unacceptable too.

- The Point P_5
This point is, independent of the value of n , represent a de Sitter phase ($w_{eff} = -1$). So, we expect that the energy density fraction of the matter must be zero ,i.e, $\Omega_m = 0$. To have a dark energy dominated era with $\Omega_m = 0$ and $\Omega_{de} = 1$, the values of n should be $n = 2$ and $n = -1$ (see Fig. 2), which leads to the critical points $(0, 0, 1, 0, 0, 0)$ and $(0, \frac{3}{2}, 4, -3, 0, -\frac{3}{2})$ respectively. The corresponding eigenvalues of these critical points are spiral stables.
- The Point P_6
Since this point has an energy density fraction of the matter, this solution can be regarded as a scaling solution. This point is, independent of the value of n , shows a matter dominated phase with $w_{eff} = 0$ and we expect that some n leads to $\Omega_m = 1$. Indeed for $n = 1$, $n = 0.143$ and $n = 2.358$ (see Fig. 2), we have $\Omega_m = 1$. In the case $n = 1$, the eigenvalues matrix diverges . However, if $n \rightarrow 1^+$ or $n \rightarrow 1^-$, this point represents a saddle matter epoch with oscillation. The values $n = 0.14$ and $n = 2.358$ leads to a saddle matter epoch with oscillation too.
- The Point P_7
In the case $n = 2$, this point is corresponding to the de Sitter point P5. Case $n = 1$, gives rise to the matter dominated era (i.e., $\Omega_m = 1$ and $w_{eff} = 0$), but its eigenvalues become singular and we cannot explain its stability. However, if $n \rightarrow 1^+$, there exists an unstable matter era with $w_{eff} = 0^-$, but if $n \rightarrow 1^-$ leads to a saddle oscillating matter epoch $w_{eff} = 0^+$.

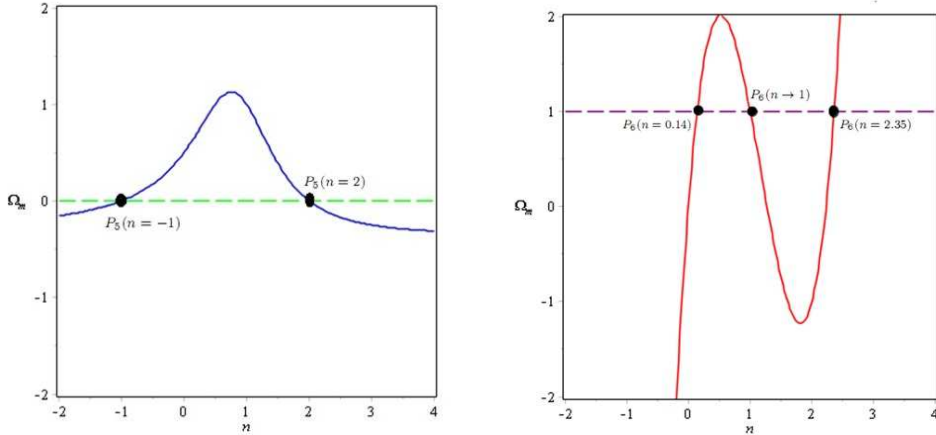


Fig. 2. (left panel) The plot of Ω_m versus n for the point P_5 . For $n = -1$ and $n = 2$, the matter density parameter vanishes which is corresponding to a de Sitter dominated phase. (right panel) behavior of Ω_m for the point P_6 . As the figure shows, for $n = 1, 0.1, 2.35$ the value of the matter density parameter is equal to 1 which corresponds to a matter dominated phase.

To have a viable cosmological model, it has to possess a matter dominated era prior to a stable accelerated expansion epoch. Now we investigate the conditions that this transition can occur from matter era to dark energy era. Point P_6 with $n = 1$, $n = 0.14$ and $n = 2.35$ is a good candidate for be in matter era because this point is a saddle oscillation point and ,once reached, it will give away to a late-time acceleration. On the other hand, the effective equation of state parameter and matter density parameter for this point are $w_{eff} = 0$ and $\Omega_m = 1$ respectively which are

compatible to a matter era. Point P_5 for $n = 2$ and $n = -1$ is a stable point with $w_{eff} = -1$ and $\Omega_{de} = 1$ therefore it is a good choice for late-time attractor. So, we study the transition from point P_6 to P_5 . If the points P_5 and P_6 had a common value of n , we could say surely that the transition is possible or not and the model for that n is cosmologically viable. But now we take one step forward and study the transition between P_6 and P_5 for different values of n . We have five possibilities of transition between these two points which study them below:

- (i) $P_6(n = 2) \rightarrow P_5(n = 2)$
In the case $n = 2$, the point P_5 , as studied above, is acceptable to be a late-time attractor. On the other hand, the point P_6 has $w_{eff} = 0$ which is ideal for the matter point but the matter density parameter is -1 which is cosmologically unacceptable.
- (ii) $P_6(n = -1) \rightarrow P_5(n = -1)$
In this case, $P_5(n = -1)$ is acceptable to be a late-time attractor but the point $P_6(n = -1)$ corresponds to $w_{eff} = 0$ and $\Omega_m = -22$. So from cosmological point of view this case is not acceptable. consequently, this transition is impossible.
- (iii) $P_6(n \rightarrow 1) \rightarrow P_5(n \rightarrow 1)$
For $P_5(n \rightarrow 1)$, although the effective equation of state parameter is corresponding to a de Sitter acceleration with $w_{eff} = -1$ but in this case, this solution has $\Omega_{de} \rightarrow 0$ which is a "dark energy era" without dark energy, that is clearly unacceptable.
- (iv) $P_6(n = 0.14) \rightarrow P_5(n = 0.14)$
The case $P_5(n = 0.14)$ has $\Omega_{de} < 1$ which is not consistent with the dark energy era as is studied above. So this transition is not acceptable too.
- (v) $P_6(n = 2.35) \rightarrow P_5(n = 2.35)$
In the case $n = 2.35$, P_6 is a saddle with oscillation point with $w_{eff} = 0$ and $\Omega_m = 1$ which is a good candidate to be in the matter era. P_5 has $w_{eff} = -1$ and $\Omega_{de} = 1.13$ which is stable spiral point with eigenvalues

$$\begin{bmatrix} -6 \\ -2.95 \\ -0.13 \\ -3.04 \\ -3.5 + 0.5i \\ -3.5 - 0.5i \end{bmatrix}$$

Since P_5 has $\Omega_{de} \approx 1$ with $w_{eff} = -1$ and is a stable spiral point, it seems that the transition $P_6(n = 2.35)$ to $P_5(n = 2.35)$ is acceptable. This issue is shown in Fig. 3.

5. Conclusion

In this work, we have considered a generalized $f(R)$ gravity theory with a non-minimal coupling between matter and curvature scalar and studied the cosmological solution in a spatially flat FLRW background. Since in this scenario we have two arbitrary functions of Ricci scalar which one of them is coupled non-minimally to the matter, the resulted equations of motion are very complicated. So to perform a phase space analysis of this model, we have considered two particular cases and in both of them can sequentially we have set one function equal to the Ricci scalar and allowed the other to be a functional form of R . To study the dynamics implied by the modified Friedmann equations, we have written the first Friedmann equation in a dimensional form and obtained the autonomous system of the first ODEs in each case. In the first case, we have assumed $f_1(R) = 2R$ and we have set favorable values for w_{eff} (the effective equation of state parameter) which is related to the several eras of the cosmic evolution and find the corresponding critical points. We have studied the system in which the universe goes through the radiation era ($w_{eff} = \frac{1}{3}$), matter era ($w_{eff} = 0$) and de Sitter era ($w_{eff} = -1$) to mimic the Λ CDM cosmology. We have found that for the radiation era, there are two critical points which one of them is a saddle point and the other is unstable. For the matter era, there exist three critical points which one of them is a saddle

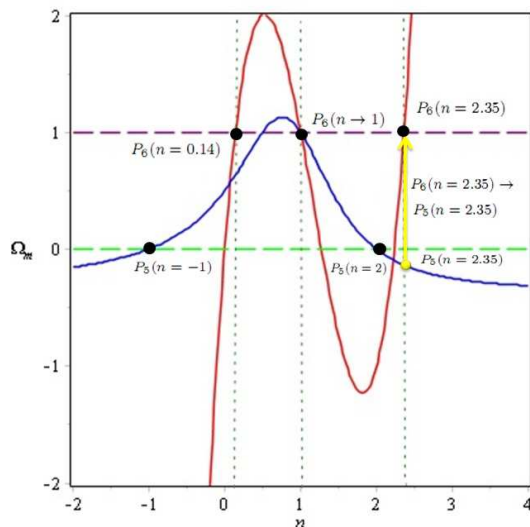


Fig. 3. The matter density parameter versus n for the critical point P_5 (blue solid line) and the point P_6 (red solid line). Intersection of the P_5 curve to the purple dashed line are corresponding to the matter dominated phase and intersection of the P_6 curve to the green dashed line are the de Sitter dominated phase.

point which corresponds to the usual matter era. In the de Sitter era we also have found three critical points which one of them is a stable point corresponding to the late time acceleration of the cosmic evolution. In the second case, we have set $f_1(R) = f(R)$ and assumed a small non-minimal coupling between the Ricci scalar and matter Lagrangian density by a small value of the coupling parameter λ . To investigate the cosmological dynamics in this case, we have assumed a power law form for $f(R)$ and found the critical points in a different manner to the previous case. By studying the stability conditions of these critical points, we have found that in the phase space of this model there are some saddle points and stable points which are corresponding to the matter era and late time acceleration era respectively. We also have investigated the conditions that the transition can occur from the matter era to dark energy era and have shown that depending on the some values of the power n , this model has a saddle matter dominated era prior to a stable accelerated expansion epoch.

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